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EXACT C*-BUNDLES

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Abstract. Kirchberg and Wassermann showed that if $\mathcal{A} = \{A, X, \pi_x : A \to A_x\}$ is a continuous C*-bundle on a locally compact Hausdorff space X with exact bundle C*-algebra A, then for any other continuous C*-bundle $\mathcal{B} = \{B, X, \pi_x : A \to B_x\}$ on X the minimal $C_0(X)$-amalgamated tensor product bundle $\mathcal{A} \otimes^{min}_{C_0(X)} \mathcal{B}$ is again continuous. In this paper it is shown conversely that this property characterises the continuous C*-bundles which have exact bundle C*-algebras when the base space X has no isolated points. For such X a corresponding result for the maximal $C_0(X)$-amalgamated tensor product of C*-bundles on X is also shown to hold, namely that $\mathcal{A} \otimes^{max}_{C_0(X)} \mathcal{B}$ is continuous for all continuous C*-bundles B on X if and only if $\mathcal{A}$ has nuclear bundle C*-algebra.

1. Introduction

Recall that a C*-bundle is a triple $\mathcal{A} = \{A, X, \pi_x : A \to A_x\}$ consisting of a C*-algebra A (the bundle algebra), a locally compact Hausdorff space X, fibre C*-algebras A_x together with $*$-epimorphisms $\pi_x : A \to A_x$ for $x \in X$ such that the family $\{\pi_x : x \in X\}$ is faithful, and an action $f \times a \to f.a$ on A of the C*-algebra $C_0(X)$ of continuous functions on X vanishing at infinity such that $\pi_x(f.a) = f(x)\pi_x(a)$ for $f \in C_0(X), a \in A, x \in X$. If the functions $x \mapsto \|\pi_x(a)\|$ are in $C_0(X)$ for all $a \in A$, the bundle $\mathcal{A}$ is said to be continuous. We shall usually denote $\mathcal{A}$ simply by $\{A, X, A_x\}$ when there is no risk of confusion. The study of continuous C*-bundles in one form or another goes back several decades; they were, for example, considered in [4] in the guise of continuous fields of C*-algebras.

More recently the closely related idea of a $C_0(X)$-algebra has been the focus of much attention. If X is a locally compact Hausdorff space and A is a C*-algebra,

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A is a $C_0(X)$-algebra if there is a non-degenerate $*$-homomorphism ι of $C_0(X)$ into the centre of the multiplier algebra $M(A)$. It is clear that if $\{A, X, A_x\}$ is a C*-bundle, then A is a $C_0(X)$-algebra. If conversely A is a $C_0(X)$-algebra, let $A_x = A/C_{0,x}(X)A$ and let $\pi_x : A \to A_x$ be the quotient map for $x \in X$, where $C_{0,x}(X) = \{f \in C_0(X) : f(x) = 0\}$. (Note that $C_{0,x}(X)A$ coincides with its closed linear span by Cohen's factorisation theorem.) Then $\{C_0(\bar{X})A, X, A_x\}$ is a C*-bundle and moreover the functions $x \mapsto \|\pi_x(a)\|$ are upper semicontinuous for $a \in C_0(\bar{X})A$.

There are various ways that continuous C*-bundles arise in practice, the simplest examples being the trivial bundles, those C*-bundles which, for a given base space X, have bundle algebra $C_0(X, A)$ for some fixed C*-algebra A, fibre A at each point of X and the obvious $C_0(X)$-algebra structure. Two related classes of continuous C*-bundles are the locally trivial and subtrivial C*-bundles, the definitions of which we recall for later reference. A C*-bundle $\mathcal{A} = \{A,X,A_x\}$ is locally trivial if, for each $x \in X$, there is a compact neighbourhood K of x such that the restriction $\mathcal{A}|_K = \{A|_K, K, A_x\}$ of $\mathcal{A}$ to K is trivial, where $A|_K$ is the image of A under the $*$-homomorphism $\oplus_{x\in K}\pi_x$. If $\mathcal{A} = \{A, X, A_x\}$ is a C*-bundle on X, a C*-bundle $\mathcal{B} = \{B, X, B_x\}$ is a C*-subbundle of $\mathcal{A}$ if B is a $C_0(X)$-subalgebra of A, that is B is a C*-subalgebra of A with $C_0(X).B \subseteq B$, B_x is a C*-subalgebra of A_x for $x \in X$ and each morphism $B \to B_x$ is the restriction to B of the corresponding morphism $A \to A_x$. It is immediate that C*-subbundles of continuous C*-bundles are continuous. A C*-bundle is subtrivial if it is a C*-subbundle of a trivial bundle. There is also a notion of local subtriviality. A C*-bundle $\mathcal{A}$ on X is locally subtrivial if for $x \in X$ there is a compact neighbourhood K of x such that $\mathcal{A}|_K$ is subtrivial. There exist continuous C*-bundles on any infinite compact Hausdorff space which are not locally subtrivial [5]. However a C*-bundle $\mathcal{A} = \{A, X, A_x\}$ on a compact metric space X with A separable such that all the A_x are nuclear or such that A is exact is always subtrivial, and is in fact $C_0(X)$-isomorphic to a C*-subalgebra of the trivial $C_0(X)$-algebra $C_0(X, \mathcal{O}_2)$ [2].

One way that discontinuous C*-bundles can arise and which is the main concern of this paper is as tensor products. Given C*-bundles $\mathcal{A} = \{A, X, \pi_x^A : A \to A_x\}$ and $\mathcal{B} = \{B, X, \pi_x^B : B \to B_x\}$ on a locally compact Hausdorff space X, minimal and maximal $C_0(X)$-tensor product bundles $\mathcal{A} \otimes_{C_0(X)}^{min} \mathcal{B}$ and $\mathcal{A} \otimes_{C_0(X)}^{max} \mathcal{B}$ can be defined. The minimal amalgamated tensor product C*-algebra $A \otimes_{C_0(X)}^{min} B$ is $(\oplus_{x\in X}\pi_x^A \otimes \pi_x^B)(A \otimes_{min} B)$ and $\mathcal{A} \otimes_{C_0(X)}^{min} \mathcal{B}$ is the C*-bundle $\{A \otimes_{C_0(X)}^{min} B, X, \pi_x^A \otimes \pi_x^B : A \otimes_{min} B \to A_x \otimes_{min} B_x\}$. Even if $\mathcal{A}$ and $\mathcal{B}$ are continuous, $\mathcal{A} \otimes_{C_0(X)}^{min} \mathcal{B}$ need not be (see [5]). If $\mathcal{A} \otimes_{C_0(X)}^{min} \mathcal{B}$ is not continuous, the C*-bundles $\mathcal{A}$ and $\mathcal{B}$

cannot both be trivial. If A is a C*-algebra and $\mathcal{B} = \{B, X, B_x\}$ is a continuous C*-bundle on the locally compact Hausdorff space X, it was shown in [5] that the minimal tensor product bundle $A \otimes \mathcal{B} = \{A \otimes B, X, A \otimes B_x\}$ is continuous for all continuous $\mathcal{B}$ if and only A is an exact C*-algebra. Moreover it was shown that for a continuous C*-bundle $\mathcal{A} = \{A, X, A_x\}$ on X with A exact, $\mathcal{A} \otimes^{min}_{C_0(X)} \mathcal{B}$ is continuous for all continuous $\mathcal{B}$ on X; and if all the A_x are exact and $\mathcal{A} \otimes^{min}_{C_0(X)} \mathcal{B}$ is continuous for all continuous $\mathcal{B}$ on X, then A is exact [5]. One of the main result of this paper is the following strengthening of this last result in the case when the base space has no isolated points. Combined with the earlier results just cited, this yields a satisfying characterisation of the continuous C*-bundles on such base spaces which have exact bundle C*-algebras and completes the circle of ideas initiated in [5].

Theorem 1.1. *Let X be a locally compact Hausdorff space with no isolated points and let $\mathcal{A} = \{A, X, A_x\}$ be a continuous C*-bundle on X such that for any continuous C*-bundle $\mathcal{B}$ on X, the minimal $C_0(X)$-tensor product $\mathcal{A} \otimes^{min}_{C_0(X)} \mathcal{B}$ is continuous. Then A is exact.*

We prove this in stages, considering successively the cases (i) $X = \widehat{\mathbb{N}}$, the one-point compactification of the natural numbers $\mathbb{N}$ with added limit point ∞ , (ii) X a compact metric space and (iii) X a compact Hausdorff space. In the case $X = \widehat{\mathbb{N}}$ we show that if the hypotheses of the theorem hold, then A_∞ is exact.

For C*-bundles $\mathcal{A} = \{A, X, \pi^A_x : A \to A_x\}$ and $\mathcal{B} = \{B, X, \pi^B_x : B \to B_x\}$ on X, the maximal amalgamated tensor product C*-algebra $A \otimes^{max}_{C_0(X)} B$ is the C*-algebra $(\oplus_{x\in X}\pi^A_x \otimes_{max} \pi^B_x)(A \otimes_{max} B)$ and the maximal tensor product C*-bundle $\mathcal{A} \otimes^{max}_{C_0(X)} \mathcal{B}$ is defined to be $\{A \otimes^{max}_{C_0(X)} B, X, \pi^A_x \otimes_{max} \pi^B_x : A \otimes_{max} B \to A_x \otimes_{max} B_x\}$ (see [1]). When $\mathcal{A}$ is the trivial bundle on X with fibre A, $\mathcal{A} \otimes^{max}_{C_0(X)} \mathcal{B}$ is denoted by $A \otimes_{max} \mathcal{B}$. It was shown in [5, section 3] that a C*-algebra A is nuclear if and only its maximal tensor product $A \otimes_{max} \mathcal{B}$ is continuous for any continuous C*-bundle $\mathcal{B}$ on $[0, 1]$. If $\mathcal{A}$ is a continuous C*-bundle on a locally compact Hausdorff space X with nuclear bundle algebra A, then each of the fibre algebras A_x is nuclear, and the maximal and minimal $C_0(X)$-tensor products $\mathcal{A} \otimes^{max}_{C_0(X)} \mathcal{B}$ and $\mathcal{A} \otimes^{min}_{C_0(X)} \mathcal{B}$ are naturally $C_0(X)$-isomorphic for any continuous C*-bundle $\mathcal{B}$ on X. Since A is exact, [5, Theorem 4.6] implies that $\mathcal{A} \otimes^{max}_{C_0(X)} \mathcal{B}$ is continuous for any continuous C*-bundle $\mathcal{B}$. Our second main result is the following converse of this in the case when the base space has no isolated points.

Theorem 1.2. *Let X be a locally compact Hausdorff space with no isolated points and let $\mathcal{A}$ be a continuous C*-bundle on X such that for any continuous C*-bundle*

$\mathcal{B}$ on X, the maximal $C_0(X)$-tensor product $\mathcal{A} \otimes^{max}_{C_0(X)} \mathcal{B}$ is continuous. Then A is nuclear.

2. Preliminaries

To simplify the notation, the minimal C*-tensor product of C*-algebras A and B will be denoted by $A \otimes B$. If $\mathcal{B} = \{B, X, B_x\}$ is a C*-bundle on a locally compact Hausdorff space X, then the minimal C*-tensor product of A and $\mathcal{B}$, the C*-bundle $\{A \otimes B, X, A \otimes B_x\}$, will be denoted by $A \otimes \mathcal{B}$, and if $\mathcal{A} = \{A, X, \pi_x : A \to A_x\}$ is another continuous C*-bundle on X, the minimal amalgamated C*-tensor product $\mathcal{A} \otimes^{min}_{C_0(X)} \mathcal{B}$ will be written $\mathcal{A} \otimes_{C_0(X)} \mathcal{B}$. For $a \in A$ and $x \in X$ we shall denote $\pi_x(a)$ by a_x. If Y is another locally compact Hausdorff space, let χ_y be the evaluation map at $y \in Y$, and for $a \in C_0(Y, A)$ let $a(y) = (\chi_y \otimes id_A)(a) \in A$. By [5, Theorem 4.5] $C_0(Y) \otimes \mathcal{A}$ is a continuous C*-bundle on X with bundle algebra $C_0(Y) \otimes A \cong C_0(Y, A)$ and fibre $C_0(Y) \otimes A_x$ at $x \in X$. For $a \in C_0(Y, A)$, $a_x \in C_0(Y, A_x)$ and

$$a_x(y) = \chi_y(a_x) = (\chi_y \otimes \pi_x)(a) = \pi_x(a(y)).$$

For each $x \in X$ the function $N(a_x) : y \mapsto \|a_x(y)\|$ is in $C_0(Y)$. Since $C_0(Y)$ is nuclear, the following result, which will be required in the proof of Proposition 3.2, is a simple consequence of [5, Theorem 4.6 (v)].

Lemma 2.1. *With $\mathcal{A}$ and Y as above, if $a \in C_0(Y) \otimes A$ then the $C_0(Y)$-valued function $x \mapsto N(a_x)$ on X is continuous.*

3. Minimal $C_0(X)$-tensor products

Proposition 3.1. *Let $\mathcal{A} = \{A, \widehat{\mathbb{N}}, A_n\}$ be a continuous C*-bundle on $\widehat{\mathbb{N}}$ such that for any unital separable continuous C*-bundle $\mathcal{B}$ on $\widehat{\mathbb{N}}$, the minimal $C(\widehat{\mathbb{N}})$-tensor product $\mathcal{A} \otimes_{C(\widehat{\mathbb{N}})} \mathcal{B}$ is continuous. Then A_∞ is exact.*

Proof. If $n_1 < n_2 < \dots$ is a sequence in $\mathbb{N}$, let $\mathcal{A}_{\{n_i\}}$ be the continuous C*-bundle $\mathcal{A}|_{\{n_1, n_2, \dots\} \cup \{\infty\}}$ on $\widehat{\mathbb{N}}$, where $\{n_1, n_2, \dots\}$ is identified with $\mathbb{N}$ via the bijective correspondence $i \leftrightarrow n_i$. Thus $\mathcal{A}_{\{n_i\}} = \{A_{\{n_i\}}, \widehat{\mathbb{N}}, \bar{A}_i\}$, where $\bar{A}_i = A_{n_i}$ and $A_{\{n_i\}}$ is the bundle algebra of $\mathcal{A}_{\{n_i\}}$. Let $\phi : \widehat{\mathbb{N}} \to \widehat{\mathbb{N}}$ be given by

$$\phi(i) = j \quad (n_{j-1} < i \leq n_j);\ \varphi(\infty) = \infty,$$

where n_0 is taken to be 0. If $\mathcal{B} = \{B, \widehat{\mathbb{N}}, B_i\}$ is a separable continuous C*-bundle on $\widehat{\mathbb{N}}$, let ${}_\phi\mathcal{B}$ be the pull-back of $\mathcal{B}$ along ϕ. Thus ${}_\phi\mathcal{B}$ is the continuous C*-bundle

$\{\bar{B}, \widehat{\mathbb{N}}, \bar{B}_i\}$, where

$$\bar{B}_i = B_j \quad (n_{j-1} < i \leq n_j), \bar{B}_\infty = B_\infty$$

and $\bar{B}$ is the C*-algebra generated by the canonical images of B and $C(\widehat{\mathbb{N}})$ in $\Pi_{i\in\widehat{\mathbb{N}}}\bar{B}_i$. Since

$$_\phi\mathcal{B}|_{\{n_1,n_2,\ldots\}\cup\{\infty\}} \cong \mathcal{B}$$

via the correspondence $i \leftrightarrow n_i$,

$$(\mathcal{A} \otimes_{C(\widehat{\mathbb{N}})} {}_\phi\mathcal{B})|_{\{n_1,n_2,\ldots\}\cup\{\infty\}} \cong \mathcal{A}_{\{n_i\}} \otimes_{C(\widehat{\mathbb{N}})} \mathcal{B}.$$

Since $\mathcal{A} \otimes_{C(\widehat{\mathbb{N}})} {}_\phi\mathcal{B}$ is continuous by hypothesis, it follows that $\mathcal{A}_{\{n_i\}} \otimes_{C(\widehat{\mathbb{N}})} \mathcal{B}$ is continuous for any separable continuous C*-bundle $\mathcal{B}$ on $\widehat{\mathbb{N}}$.

2. Let $\mathcal{B}$ be a unital separable continuous C*-bundle on $\widehat{\mathbb{N}}$, let $x = \sum_{k=1}^n a_k \otimes b_k$ be in the algebraic tensor product $A_\infty \odot B$ and let $\varepsilon > 0$. For each k there is an $\bar{a}_k \in A$ such that $\pi_\infty(\bar{a}_k) = a_k$. Since $\mathcal{A} \otimes B_l$ is continuous for $l \in \mathbb{N}$,

$$\lim_{i\to\infty} \|(\pi_i \otimes \pi_l)(\sum_k \bar{a}_k \otimes b_k)\| = \|\sum_k a_k \otimes \pi_l(b_k)\|.$$

It follows that we can find by induction a sequence $n_1 < n_2 \ldots$ in $\mathbb{N}$ such that for each l and $i \geq n_l$,

$$\big|\, \|\sum_k a_k \otimes \pi_l(b_k)\| - \|(\pi_i \otimes \pi_l)(\sum_k \bar{a}_k \otimes b_k)\| \,\big| < \varepsilon/2. \tag{1}$$

Since by part 1 of the proof $\mathcal{A}_{\{n_i\}} \otimes_{C(\widehat{\mathbb{N}})} \mathcal{B}$ is continuous,

$$\lim_{l\to\infty} \|(\pi_{n_l} \otimes \pi_l)(\sum_k \bar{a}_k \otimes b_k)\| = \|\sum_k a_k \otimes \pi_\infty(b_k)\|,$$

which implies that there is an $N \in \mathbb{N}$ such that for $l \geq N$,

$$\big|\, \|(\pi_{n_l} \otimes \pi_l)(\sum_k \bar{a}_k \otimes b_k)\| - \|\sum_k a_k \otimes \pi_\infty(b_k)\| \,\big| < \varepsilon/2 \tag{2}$$

Combining (1) and (2),

$$\big|\, \|\sum_k a_k \otimes \pi_l(b_k)\| - \|\sum_k a_k \otimes \pi_\infty(b_k)\| \,\big| < \varepsilon.$$

for $l \geq N$. Since ε is arbitrary, the function

$$l \to \|\sum_k a_k \otimes \pi_l(b_k)\|$$

is continuous on $\widehat{\mathbb{N}}$. This implies that $A_\infty \otimes \mathcal{B}$ is continuous, since $A_\infty \odot B$ is dense in $A_\infty \otimes B$, which implies in turn that A_∞ is exact, by [5, Theorem 4.5]. □

In what follows CA will denote the cone $A \otimes C_0((0,1])$ of the C*-algebra A. If $\mathcal{A}$ is a continuous C*-bundle on a compact Hausdorff space X, the cone $C\mathcal{A}$ of $\mathcal{A}$ is the C*-bundle $\mathcal{A} \otimes C_0((0,1])$ on X. By [5, Theorem 4.5], $C\mathcal{A}$ is necessarily continuous, since $C_0((0,1])$ is nuclear.

Proposition 3.2. *Let X be a compact metric space, Y a nonempty compact subset of X and $\mathcal{A} = \{A, Y, A_y\}$ a continuous C*-bundle on Y. There exists a continuous C*-bundle $\bar{\mathcal{A}} = \{\bar{A}, X, \bar{A}_x\}$ on X such that*

$$\bar{\mathcal{A}}|_Y \cong C\mathcal{A}.$$

If $\mathcal{A}$ is separable, $\bar{\mathcal{A}}$ can be chosen separable.

Proof. 1. **Construction.** Let $d(.,.)$ be the metric on X and let ϕ be the real function on $(0,\infty)$ given by

$$\phi(t) = \begin{cases} 1 & (t \leq 1) \\ 2-t & (1 < t \leq 2) \\ 0 & (t > 2). \end{cases}$$

For $x \in X \setminus Y$ let g_x be the real function on Y given by $g_x(y) = d(x,y)/d(x,Y)$ and let $\phi_x = \phi \circ g_x$. Let $\pi_{y,t} : CA \to CA_y$ be the $*$-homomorphism given by

$$\pi_{y,t}(f \otimes a) = f_t \otimes \pi_y(a) \quad (f \in C_0((0,1]), a \in A),$$

where $f_t(s) = f(st)$ for $f \in C_0((0,1])$, $s,t \in [0,1]$, and $C_0((0,1])$ is identified with the ideal $\{f \in C([0,1]) : f(0) = 0\}$ of $C([0,1])$, so that $f(0)$ is defined and equal to 0. For $x \in X \setminus Y$ let $\bar{\pi}_x : CA \to \oplus_{y \in Y} CA_y$ be the $*$-homomorphism given by

$$\bar{\pi}_x(c) = \oplus_{y \in Y} \pi_{y,\phi_x(y)}(c).$$

For $y \in Y$ let $\bar{\pi}_y : CA \to CA_y$ be the $*$-homomorphism

$$id_{C((0,1])} \otimes \pi_y : f \otimes a \mapsto f \otimes \pi_y(a).$$

The fibre algebras $\bar{A}_x$ of the bundle $\bar{\mathcal{A}}$ are given by

$$\bar{A}_x = \begin{cases} CA_x & (x \in Y) \\ \bar{\pi}_x(CA) & (x \in X \setminus Y). \end{cases}$$

The bundle C*-algebra $\bar{A}$ is the $C(X)$-subalgebra of $\Pi_{x \in X} \bar{A}_x$ generated by the image of CA under the embedding $a \mapsto (\bar{\pi}_x(a))_{x \in X}$, where the action of $C(X)$ is given by

$$f.(\bar{\pi}_x(a))_{x \in X} = (f(x)\bar{\pi}_x(a))_{x \in X}.$$

Thus $\bar{A}$ is the closed linear span of $C(X).CA$ (and actually coincides with the set $C(X).CA$ since $1 \in C(X)$). The natural extension of $\bar{\pi}_x$ to $\bar{A}$ will again be denoted by $\bar{\pi}_x$. It is immediate that $\bar{\pi}_x : \bar{A} \to \bar{A}_x$ is surjective for $x \in X$, and that $\bar{A}$ is a $C(X)$-algebra, so that $\{\bar{A}, X, \bar{A}_x\}$ is a C*-bundle. Moreover the restriction of $\bar{\mathcal{A}}$ to Y is $C\mathcal{A}$.

2. **Continuity.** 1. Let $x \in X \setminus Y$. If $x' \in X \setminus Y$ then

$$d(x, Y) \leq d(x, x') + d(x', Y),$$

so that $d(x', Y) > d(x, Y)/2$ if $d(x, x') < d(x, Y)/2$ and for $y \in Y$,

$$\begin{aligned}
g_{x'}(y) - g_x(y) &= \frac{d(x', y)}{d(x', Y)} - \frac{d(x, y)}{d(x, Y)} \\
&= d(x', y)(d(x, Y) - d(x', Y)) + d(x', Y)(d(x', y) - \frac{d(x, y))}{d(x', Y)d(x, Y)} \\
&\leq d(x', x)(d(x', y) + \frac{d(x', Y))}{d(x', Y)d(x, Y)} \\
&\leq 3d(x', x)d(X)/d(x, Y)^2,
\end{aligned}$$

where $d(X) = \sup_{x_1, x_2 \in X} d(x_1, x_2)$. Thus $g_{x'} \to g_x$ uniformly on Y as $x' \to x$ and, since ϕ is uniformly continuous, $\phi_{x'} \to \phi_x$ uniformly on Y as $x' \to x$. It follows that for $a \in \bar{A}$, $\|\bar{\pi}_{x'}(a) - \bar{\pi}_x(a)\| \to 0$ as $x' \to x$, where $\bar{\pi}_{x'}(a)$ and $\bar{\pi}_x(a)$ are both regarded as elements of $\oplus_{y \in Y} CA_y$. This shows that $\bar{\mathcal{A}}$ is continuous at x.

2. Let $y_0 \in Y$. If $a \in CA$ and $\varepsilon > 0$, since $C\mathcal{A}$ is continuous on Y there is by Lemma 2.1 a $\delta > 0$ such that for $y \in Y$,

$$d(y, y_0) < \delta \Rightarrow \| N(a_y) - N(a_{y_0}) \|_\infty < \varepsilon,$$

where $a_x = \bar{\pi}_x(a)$ for $x \in X$. Suppose that $x \in X \setminus Y$ satisfies $d(x, y_0) < \delta/4$. Then

$$\begin{aligned}
g_x(y) \leq 2 \quad &\Rightarrow \quad d(x, y) \leq 2d(x, Y) \leq \delta/2 \\
&\Rightarrow \quad d(y, y_0) \leq d(y, x) + d(x, y_0) < \delta \qquad (3)\\
&\Rightarrow \quad \| N(a_y) - N(a_{y_0}) \|_\infty < \varepsilon \\
&\Rightarrow \quad \big| \|\pi_{y, \phi_x(y)}(a)\| - \|\pi_{y_0, \phi_x(y)}(a)\| \big| < \varepsilon.
\end{aligned}$$

Now

$$\|a_x\| = \sup_{d(y, y_0) < \delta} \|\pi_{y, \phi_x(y)}(a)\| \qquad \text{by (3)}$$

and

$$\|a_{y_0}\| = \|\pi_{y_0, 1}(a)\| = \sup_{d(y, y_0) < \delta} \|\pi_{y_0, \phi_x(y)}(a)\|,$$

from which it follows that

$$|\,\|a_x\| - \|a_{y_0}\|\,| < \varepsilon.$$

If $y \in Y$ satisfies $d(y, y_0) < \delta/4$, then

$$\|\,N(a_y) - N(a_{y_0})\,\|_\infty < \varepsilon,$$

which implies that $|\,\|a_y\| - \|a_{y_0}\|\,| < \varepsilon$. Thus for any $x \in X$ satisfying $d(x, y_0) < \delta/4$,

$$|\,\|a_x\| - \|a_{y_0}\|\,| < \varepsilon,$$

which establishes the continuity of $\|a_x\|$ at y_0. The continuity of $\bar{\mathcal{A}}$ now follows straightforwardly, since, by construction, for arbitrary $a \in \bar{A}$, $x_0 \in X$ and $\varepsilon > 0$ there exist an open neighbourhood U of x and an $a' \in CA$ such that $\|a_x - a'_x\| < \varepsilon$ for $x \in U$. □

Proposition 3.3. *Let X be a compact metric space. If $\mathcal{A} = \{A, X, A_x\}$ is a continuous C*-bundle on X such that $\mathcal{A} \otimes_{C(X)} \mathcal{B}$ is continuous for any continuous C*-bundle $\mathcal{B}$ on X, then A_x is exact for any limit point $x \in X$.*

Proof. If $x_0 \in X$ is a limit point, let $\{x_i\}$ be a sequence of distinct points in $X \setminus \{x_0\}$ with limit x_0 and let $\mathcal{B} = \{B, \widehat{\mathbb{N}}, B_n\}$ be a separable continuous C*-bundle on $\widehat{\mathbb{N}}$. Identifying the topological spaces $\widehat{\mathbb{N}}$ and $\{x_1, x_2, \ldots\} \cup \{x_0\}$ in the obvious way, there is a separable continuous bundle $\bar{\mathcal{B}}$ on X such that $\bar{\mathcal{B}}|_{\{x_1,x_2,\ldots\}\cup\{x_0\}} \cong C\mathcal{B}$ by Proposition 3.2. Since

$$(\mathcal{A} \otimes_{C(X)} \bar{\mathcal{B}})|_{\{x_1,x_2,\ldots\}\cup\{x_0\}} \cong \mathcal{A}|_{\{x_1,x_2,\ldots\}\cup\{x_0\}} \otimes_{C(\hat{\mathbb{N}})} C\mathcal{B} \cong C(\mathcal{A}|_{\{x_1,x_2,\ldots\}\cup\{x_0\}} \otimes_{C(\hat{\mathbb{N}})} \mathcal{B}),$$

the continuity of $\mathcal{A} \otimes_{C(X)} \bar{\mathcal{B}}$ implies that of $C(\mathcal{A}|_{\{x_1,x_2,\ldots\}\cup\{x_0\}} \otimes_{C(\widehat{\mathbb{N}})} \mathcal{B})$. Letting $z \in C((0,1])$ be the function given by $z(t) = t$, the map $a \mapsto a \otimes z$ from $\mathcal{A}|_{\{x_1,x_2,\ldots\}\cup\{x_0\}} \otimes_{C(\hat{\mathbb{N}})} \mathcal{B}$ to $C((\mathcal{A}|_{\{x_1,x_2,\ldots\}\cup\{x_0\}} \otimes_{C(\hat{\mathbb{N}})} \mathcal{B})$ is a completely positive $C(\hat{\mathbb{N}})$-isometry. This implies that $\mathcal{A}|_{\{x_1,x_2,\ldots\}\cup\{x_0\}} \otimes_{C(\widehat{\mathbb{N}})} \mathcal{B}$ is continuous. Since $\mathcal{B}$ is arbitrary, A_{x_0} is exact, by Proposition 3.1. □

Proof of Theorem 1.1. We first consider the case X compact. Let $x_0 \in X$. If A_{x_0} is finite dimensional, it is trivially exact. If on the other hand A_{x_0} is infinite dimensional, let C be a separable infinite dimensional C*-subalgebra of A_{x_0}. We show that C is exact.

For $a \in A$ let $N(a)$ denote the continuous real function $x \mapsto \|\pi_x(a)\|$ on X. Let D be a separable C*-subalgebra of A such that $\pi_{x_0}(D) = C$. Then the unital abelian C*-algebra $C^*(N(D), 1)$ is separable and its spectrum is compact and second countable, hence metrizable, and a quotient of X. To make this explicit, let $\{f_1, f_2, \ldots\}$ be a dense sequence in $C^*(N(D), 1) \setminus \{0\}$ and let an equivalence relation $\sim$ on X be defined by $x \sim y \Leftrightarrow f_n(x) = f_n(y)$ for all n. Writing $\bar{x}$ for the equivalence class of x and $\bar{X}$ for the set $\{\bar{x} : x \in X\}$ of equivalence classes, a metric on $\bar{X}$ is defined by

$$d(\bar{x}, \bar{y}) = \sum_{n=1}^{\infty} 2^{-n} \|f_n\|^{-1} |f_n(x) - f_n(y)|.$$

It is routine to verify that with the obvious identifications, the spectrum of $C^*(N(D))$ is $\bar{X}$ with the topology coming from this metric. Moreover the map $\phi : x \mapsto \bar{x}$ from X to $\bar{X}$ is continuous and surjective. There are two cases to distinguish, according to whether $\bar{x}_0$ is an isolated point of $\bar{X}$ or not.

(a) $\bar{x}_0$ **isolated.** In this case the subset $X_0 = \phi^{-1}(\bar{x}_0)$ of X is open, compact and infinite, since x_0 is not an isolated point of X. Let $\mathcal{B}$ be a continuous C*-bundle on X_0, let X_2 be the compact set $X \setminus X_0$ and let $\mathcal{B}' = \mathcal{B} \oplus \mathcal{C}(X_2)$, where $\mathcal{C}(X_2)$ is the trivial C*-bundle on X_2 with fibre $\mathbb{C}$. Then $\mathcal{B}'$ is a continuous C*-bundle on X and by hypothesis $\mathcal{A} \otimes_{C(X)} \mathcal{B}'$ is continuous. Hence $(\mathcal{A}|_{X_0}) \otimes_{C(X_0)} \mathcal{B} = (\mathcal{A} \otimes_{C(X)} \mathcal{B}')|_{X_0}$ is continuous. This implies in particular that $N(b)$ is continuous on X_0 for $b \in D \otimes_{C(X)} B'$. However for $d \in D$, $\|\pi_x(d)\| = \|\pi_{x_0}(d)\|$ for $x \in X_0$, which implies that $\pi_x(D) \cong C$ for $x \in X_0$ via the isomorphism $\pi_x(d) \mapsto \pi_{x_0}(d)$, so that the restrictions of the sections of D to X_0 are identified with the constant C-valued functions on X_0. Thus the algebra of sections of $(\mathcal{A}|_{X_0}) \otimes_{C(X_0)} \mathcal{B}$ coming from $D \otimes_{C_0(X)} B'$ is isomorphic to $C \otimes \mathcal{B}$ and is continuous. Since $\mathcal{B}$ is arbitrary and X_0 is infinite, X_0 has a limit point and C is exact by [3, Corollary 4].

(b) $\bar{x}_0$ **is a limit point.** If $x \sim x'$, so that $\bar{x} = \bar{x}'$, then $\|\pi_x(d)\| = \|\pi_{x'}(d)\|$ for $d \in D$. Thus $\pi_x(D) \cong \pi_{x'}(D)$, and we can identify $\pi_x(D)$ and $\pi_{x'}(D)$, and also π_x and $\pi_{x'}$. With these identifications, for $\bar{x} \in \bar{X}$ let $\bar{D}_{\bar{x}} = \pi_x(D)$, $\pi_{\bar{x}}(d) = \pi_x(d)$ and $\bar{D} = C^*(C(\bar{X})D)$, where the action of $C(\bar{X})$ on D is given by

$$f.d = (f \circ \phi)d$$

for $f \in C(\bar{X})$ and $d \in D$. The triple $\bar{\mathcal{D}} = \{\bar{D}, \bar{X}, \bar{D}_{\bar{x}}\}$ is a continuous C*-bundle on $\bar{X}$. For $n = 1, 2, \ldots$ let $\bar{f}_n$ be the function on $\bar{X}$ given by $\bar{f}_n(\bar{x}) = f_n(x)$. By construction, $\bar{f}_1, \bar{f}_2, \ldots$ are continuous.

Let $\mathcal{B} = \{B, \bar{X}, B_{\bar{x}}\}$ be a continuous C*-bundle on $\bar{X}$ and let ${}_\phi\mathcal{B}$ be the pull-back to X. By hypothesis $\mathcal{A} \otimes_{C(X)} {}_\phi\mathcal{B}$ is continuous, which implies that if $d \in \bar{D} \otimes_{C(\bar{X})} B$, and f is the non-negative real function on $\bar{X}$ defined by

$$f(\bar{x}) = \|\pi_{\bar{x}}(d)\|$$

for $x \in X$, then the function $f \circ \phi : x \mapsto f(\bar{x})$ is continuous. To see that f is itself continuous, let $x_1, x_2, \ldots$ be a sequence in X such that $\bar{x}_i \to \bar{x}_0$ in $\bar{X}$ and let $\{\bar{x}_{i_1}, \bar{x}_{i_2}, \ldots\}$, where $i_1 < i_2 < \ldots$, be a subsequence of $\{\bar{x}_1, \bar{x}_2, \ldots\}$ such that $f(\bar{x}_{i_j})$ converges as $j \to \infty$, with $\alpha = \lim_{j\to\infty} f(\bar{x}_{i_j})$. By the compactness of X, $\{x_{i_1}, x_{i_2}, \ldots\}$ has a limit point x' in X. From the continuity of f_n and $\bar{f}_n$,

$$f_n(x_i) = \bar{f}_n(\bar{x}_i) \to \bar{f}_n(\bar{x}_0) = f_n(x_0)$$

as $i \to \infty$, which implies that $f_n(x') = f_n(x_0)$ and $x' \sim x_0$. Since the function $x \mapsto f(\bar{x})$ is continuous, $f(\bar{x}') = f(\bar{x}_0)$ is a limit point of $\{f(\bar{x}_{i_1}), f(\bar{x}_{i_2}), \ldots\}$, which converges to α. Thus $\alpha = f(\bar{x}_0)$, and every convergent subsequence of $\{f(\bar{x}_1), f(\bar{x}_2), \ldots\}$ converges to $f(\bar{x}_0)$. This implies that $\lim_{i\to\infty} f(\bar{x}_i)$ exists and equals $f(\bar{x}_0)$, that is, f is continuous at $\bar{x}_0$. Thus $\bar{\mathcal{D}} \otimes_{C(\bar{X})} \mathcal{B}$ is continuous at $\bar{x}_0$ for all continuous C*-bundles $\mathcal{B}$ on $\bar{X}$. By Proposition 3.3, $C \cong D_{\bar{x}_0}$ is exact.

Since a C*-algebra is the inductive limit of its net of separable C*-subalgebras and an inductive limit of exact C*-algebras is itself exact, A_x is exact for all $x \in X$. It now follows by [5, Theorem 4.6] that A is exact.

In the general case, if X is not compact, let $\widehat{X}$ be the one-point compactification of X, with added point ω, and let $\hat{\mathcal{A}} = \{A, \widehat{X}, A_x\}$ be the natural extension of $\mathcal{A}$ to $\widehat{X}$, so that the fibre A_ω at ω is $\{0\}$ and $\pi_\omega = 0$. Then $\hat{\mathcal{A}}$ is continuous. If $\mathcal{B}$ is a continuous C*-bundle on $\widehat{X}$ and $C_0(X)$ is identified with the ideal of functions in $C(\widehat{X})$ vanishing at ω, then $\mathcal{B}_0 = \{C_0(X).B, X, A_x\}$ is a continuous C*-bundle on X. The amalgamated tensor product $\mathcal{A} \otimes^{min}_{C_0(X)} \mathcal{B}_0$ is continuous by hypothesis and its natural extension to $\widehat{X}$ coincides with $\hat{\mathcal{A}} \otimes^{min}_{C(\widehat{X})} \mathcal{B}$. Thus $\hat{\mathcal{A}} \otimes^{min}_{C(\widehat{X})} \mathcal{B}$ is continuous for any continuous C*-bundle $\mathcal{B}$ on $\widehat{X}$. By the proof given for the compact case, A is exact. $\square$

4. Maximal $C_0(X)$-tensor products

To prove Theorem 1.2, we prove the following more general result.

Theorem 4.1. *Let X be a locally compact Hausdorff space and let $\mathcal{A}$ be a continuous C*-bundle on X. Let $x_0 \in X$ be a limit point. Then $\mathcal{A} \otimes^{max}_{C_0(X)} \mathcal{B}$ is continuous at x_0 for all continuous C*-bundles $\mathcal{B}$ on X if and only if A_{x_0} is nuclear. Moreover if A_{x_0} is nuclear, then $\mathcal{A} \otimes^{min}_{C_0(X)} \mathcal{B}$ is continuous at x_0 for all continuous C*-bundles $\mathcal{B}$.*

Proof. Let x_0 be a limit point of X and assume that A_{x_0} is nuclear. Let $\mathcal{B} = \{B, X, B_x\}$ be another continuous C*-bundle on X and let $a \in A \otimes^{max}_{C_0(X)} B$. Given $\varepsilon > 0$, since the function $x \mapsto \|a_x\|_{max}$ is upper semicontinuous [5, Lemma 2.3], there is an open neighbourhood U of x such that

$$\|a_x\|_{max} \leq \|a_{x_0}\|_{max} + \varepsilon$$

for $x \in U$. Let $\tilde{a}$ be the image of a in $A \otimes^{min}_{C_0(X)} B$ under the canonical $C_0(X)$-homomorphism from $A \otimes^{max}_{C_0(X)} B$ to $A \otimes^{min}_{C_0(X)} B$. Then $\|\tilde{a}_{x_0}\|_{min} = \|a_{x_0}\|_{max}$ since A_{x_0} is nuclear. Since the function $x \mapsto \|\tilde{a}_x\|_{min}$ is lower semicontinuous [5, Prop. 4.9], there is an open neighbourhood V of x_0 such that

$$\|\tilde{a}_x\|_{min} \geq \|\tilde{a}_{x_0}\|_{min} - \varepsilon$$

for $x \in V$. Thus if $x \in U \cap V$,

$$\|a_{x_0}\| - \varepsilon \leq \|\tilde{a}_x\|_{min} \leq \|a_x\|_{max} \leq \|a_{x_0}\| + \varepsilon.$$

Since ε is arbitrary, this implies that the functions $x \mapsto \|a_x\|_{max}$ and $x \mapsto \|\tilde{a}_x\|_{min}$ are continuous at x_0. Thus $\mathcal{A} \otimes^{max}_{C_0(X)} \mathcal{B}$ and $\mathcal{A} \otimes^{min}_{C_0(X)} \mathcal{B}$ are continuous at x_0.

For the converse, assume that A_y is not nuclear at a limit point y of X. Then there exist a Hilbert space H, a C*-algebra $C \subset B(H)$ and a finite sum $t = \sum r_k \otimes s_k \in A_y \odot C$ such that $\|t\|_{A_y \otimes_{max} B(H)} < \|t\|_{A_y \otimes_{max} C}$ (see [5, section 3]). Using functional calculus, we can assume that $\|t\|_{A_y \otimes_{max} B(H)} < 1$ and $\|t\|_{A_y \otimes_{max} C} = 2$. Letting $B_x = B(H)$ for x in $X \backslash \{y\}$, $B_y = C$ and $B = \{b \in C(X, B(H)) : b_y \in C\}$, $\mathcal{B} = \{B, X, B_x\}$ is a continuous C*-bundle on X. Let $\bar{r}_k \in A$ and $\bar{s}_k \in B$ be liftings of r_k and s_k, respectively. Then $\bar{t} = \sum \bar{r}_k \otimes \bar{s}_k$ is in both $D = A \otimes^{max}_{C_0(X)} B$ and $A \otimes^{max}_{C_0(X)} C_0(X, B(H)) = A \otimes_{max} B(H)$. Regarding $\bar{t}$ as a section of the bundle $\mathcal{A} \otimes_{max} B(H)$ with projection $\bar{t}_x$ at x, the function $x \mapsto \|\bar{t}_x\|_{A_x \otimes_{max} B(H)}$ is upper semi-continuous at y, so that for some neighbourhood U of y, $x \in U \Rightarrow \|\bar{t}_x\|_{A_x \otimes_{max} B(H)} \leq 3/2$ since $\|\bar{t}_y\|_{A_y \otimes_{max} B(H)} < 1$. Regarding $\bar{t}$ as a section of the bundle $\mathcal{A} \otimes^{max}_{C_0(X)} \mathcal{B}$, for $x \in U \setminus \{y\}$, $\|\bar{t}_x\|_{A_x \otimes_{max} B_x} = \|t\|_{A_x \otimes_{max} B(H)} \leq 3/2$ but $\|\bar{t}_y\|_{A_y \otimes_{max} B_y} = \|t\|_{A_y \otimes_{max} C} = 2$, which shows that $\mathcal{A} \otimes^{max}_{C_0(X)} \mathcal{B}$ is not continuous at y. □

5. Further results and questions

1. Corollary 4 of [3], which was used in the proof of Theorem 1.1, states that, given an infinite compact Hausdorff space X, a C*-algebra A is exact if and only if $A \otimes \mathcal{B}$ is a continuous C*-bundle for any continuous C*-bundle $\mathcal{B}$ on X. If X is metrizable, this is a special case of Proposition 3.3, and in fact the proof given in [3] involves similar ideas to those used here. For general non-metrizable X, there

seems to be no way to prove this result without using the method of [3], which we recall briefly for completeness. Using Urysohn's Lemma, there exists a subset $\{x_1, x_2, \ldots\}$ of distinct points of X with a limit point x_0 not in the set and a continuous function $f : X \to [0,1]$ such that $f(x_1) < f(x_2) < \ldots$ and $f(x_0) = 1$. Then $f(X)$ is a compact subset of $[0,1]$ containing 1 as a limit point. If $\mathcal{B}$ is a continuous C*-bundle on $f(X)$, then the pull-back ${}_f\mathcal{B}$ is a continuous C*-bundle on X. If $A \otimes {}_f\mathcal{B}$ is continuous on X, then $A \otimes \mathcal{B}$ is continuous on $f(X)$. By the result for the metrizable case, A is exact.

2. In Proposition 3.2 can the cone $C\mathcal{A}$ be replaced by $\mathcal{A}$ itself, that is, is every continuous C*-bundle on a closed subset of a compact metric (or, more generally, compact Hausdorff) space the restriction of a continuous bundle on the whole space? We have been unable to settle this question in general, though the following essentially straightforward special cases are worth noting.

Proposition 5.1. *Let X be a compact Hausdorff space, let Y be a non-empty closed subset of X and let $\mathcal{A}$ be a continuous C*-bundle on Y. Then there exists a continuous C*-bundle $\bar{\mathcal{A}}$ on X such that $\bar{\mathcal{A}}|_Y \cong \mathcal{A}$ if (i) Y is finite, or (ii) $\mathcal{A}$ is separable and exact.*

Proof. In either case $\mathcal{A}$ is a C*-subbundle of a trivial bundle on Y with fibre a C*-algebra B. If Y is finite, then B can be taken to be the (finite) direct sum of the fibres of $\mathcal{A}$ at the points of Y. If $\mathcal{A}$ is separable and exact, then $C(Y)$ is separable, which implies that Y is metrizable, and hence, by [2, Appendix], that $\mathcal{A}$ is $C(Y)$-isomorphic to a C*-subbundle of the trivial bundle on Y with fibre $\mathcal{O}_2$. Since the restriction map $f \to f|_Y$ from $C(X)$ to $C(Y)$ is surjective, the corresponding restriction homomorphism $\phi : C(X, B) \to C(Y, B)$ is surjective and a suitable C*-bundle $\bar{\mathcal{A}}$ on X extending $\mathcal{A}$ is obtained by taking $\phi^{-1}(A)$ as the bundle algebra, where A is the bundle algebra of $\mathcal{A}$. □

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